Volumen

3

ORBITS IN THE SOLAR SYSTEM

Kepler's Laws, Conics, Orbital motion

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Astronomy Workshop

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1

Kepler's Laws

Johannes Kepler (1571-1630) was a contemporary of Galileo and had accepted the Copernican Doctrine from youth. He was totally convinced that there was a clean mathematical formulation behind the planetary motion. Kepler noticed that the farther from the Sun the slower the planets move. This led him to suggest that planets were kept in motion by the action of a force exerted from the Sun; such a force should decrease with the distance from the Sun.

Kepler contacted Tycho Brahe and worked as his assistant investigator. Tycho assigned the study of the orbit of Mars to Kepler. This was the most difficult to adjust with the models available at that time. Kepler's originality lay in trying to solve the problem not by adding more eccentric but by adjusting it to a simple geometric figure. After various unsuccessful attempts, he realised that the orbit seemed to be compressed in one direction until he finally found that it fitted to an ellipse with the Sun at one of its foci (Kepler's First Law). Kepler also studied the variation of the velocity of the planet, following Aristoteles dynamical theory, he hypothesized that variations in the velocity of the planets are caused by a change in the force acting on them. Therefore, it was plausible that the force that came from the Sun would vary inversely with the distance so that the planet would sweep over equal areas in equal intervals of time (Kepler's Second Law).

In 1609, after 8 years of work, Kepler published his First and Second Laws in the book entitled: "New Astronomy". Kepler's Laws sum up the basis of the kinematics of planetary motion and as such, can be (and are) used to derive the positions of the planets. They also correspond to the exact solution of the *two-body problem*.

The two-body problem

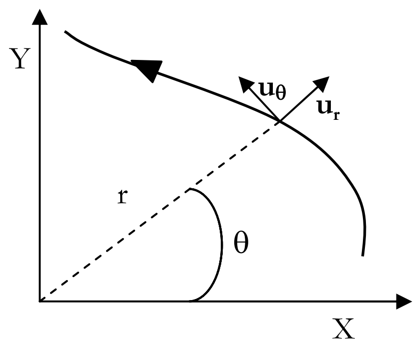
The *two-body problem* is the mathematical study/formulation of the gravitational interaction between two masses. The modern formulation of the problem is very simple; Newton's third law is written at the baricenter (or centre of mass) of the system and trajectories are determined by the solution of a very simple differential equation:

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} = -\frac{G(M+m)}{r^3}\vec{r} \qquad [1]$$

where "r" represents the distance between two interacting bodies ($\vec{r} = \vec{r}_M - \vec{r}_m$) with masses "$m$" and "$M$", and "$G$" is the gravitational constant.. The trajectories of each mass with respect to the instantaneous centre of mass are given by

$$\vec{r}_m = -\frac{M}{M+m}\vec{r} \qquad \vec{r}_M = \frac{m}{M+m}\vec{r}$$

The equation is solved by writing the vector $\vec{r}$ and its derivates in a co-moving reference frame (**u_θ, u_r**) instead of the standard (X,Y) system. The relation between both of them is shown in the figure[1]



In this system, equation [1] becomes:

$$(\ddot{r} - r\dot{\theta}^2)\vec{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{u}_\theta = -G\frac{(M+m)}{r^2}\vec{u}_r$$

so,

$$\boxed{\begin{aligned} &\ddot{r} - r\dot{\theta}^2 = -G\frac{(M+m)}{r^2} \\ &2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \end{aligned}}$$

The second equation is the mathematical formulation of the angular momentum conservation (or Kepler's Second Law). Integration requires the introduction of a conserved constant, $\boldsymbol{h}$, the angular momentum per unit of mass, so, $r^2\dot{\theta} = h$. The solution to the first equation is a conic:

$$\frac{1}{r} = A\cos(\theta - \theta_0) + \frac{G(M+m)}{h^2} \qquad [2]$$

where **A** and $\boldsymbol{\theta_0}$ are integration constants. Equation [2] corresponds to a conic with eccentricity (**e**) and semi-major axis (**a**) :

$$e = \frac{A}{G(M+m)/h^2} = \sqrt{\frac{2\varepsilon h^2}{[G(M+m)]^2} + 1}$$

$$a = \frac{G(M+m)}{2\varepsilon}$$

---

[1] $\vec{r}$ marks the instantaneous modulus and orientation with respect to the X axis of $(\vec{r}_M - \vec{r}_m)$.

where **ε** represents the conserved energy (per unit mass):

$$\varepsilon = \frac{\dot{\vec{r}}^{\,2}}{2} - G\frac{M+m}{r}$$

Thus, the two integration constants (**h,ε)** are related to the two conserved physical magnitudes (angular momentum and energy). Energy controls the size of the orbit, while the eccentricity is controlled by a combination of the two.

Conics and Orbits:

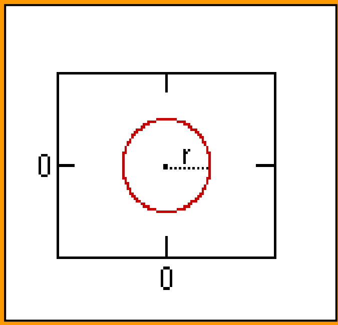
The relationship between the geometry of the orbit and the fundamental physical parameters of the problem can be summed up in the following table.

e=0	Circular orbit	$\varepsilon = \varepsilon_{\min} = -\frac{[G(M+m)]^2}{2h^2}$
0<e<1	Elliptic orbit	$\varepsilon_{\min} < \varepsilon < 0$
e=1	Parabolic orbit	$\varepsilon = 0$
e>1	Hyperbolic orbit	$\varepsilon > 0$

If **ε<0,** orbits are bound and both objects are tight together unless one of them is given some extra energy by another mechanism. If **ε>0,** orbits are unbound and the interacting bodies can escape from their mutual gravitational attraction. Parabolic and circular orbits are limiting cases that cannot be measured in Nature, as this would imply an infinite precision, which does not exist. The orbits of Solar System bodies (i.e., bodies trapped by the gravity of the Sun) are elliptic. The orbits of bodies that escape from the Solar System are hyperbolic. Space probes can use hyperbolic orbits (using the gravity of nearby massive planets, for example, Jupiter) to minimize the fuel spent in trajectories which must reach objects that are further away.

SUMMARY OF CONICS

Circle:



Locus of the points in a plane whose distance from a fixed point is constant.

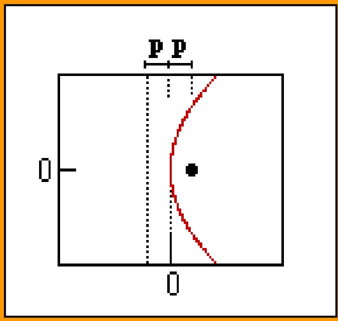
Equation and parameters:

$$x^2 + y^2 = r^2$$

r = the radius of the circle

Eccentricity: 0

Parabola:



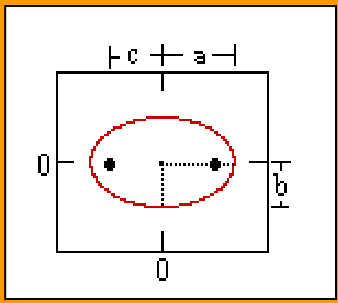
Set of all points in a plane such that each point in the set is equidistant from a line called the directrix and a fixed point called the focus. Equation and parameters::

$$y^2 = 4px$$

p = the distance from the vertex to the focus (or the directrix)

Eccentricity:: 1

Ellipse:



Locus of the points the sum of whose distance from two fixed points is constant.

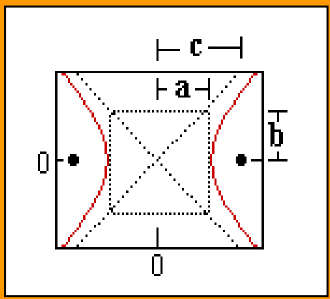
Equation and parameters::

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a = major radius (= 1/2 the longitude of the major axis)
b = minor radius (= 1/2 the longitude of the minor axis)
c = the distance from the centre to the focus.
$a^2 - b^2 = c^2$

Eccentricity: between 0 and 1

Hyperbola:



Locus of the points the difference of whose distance from two fixed points is constant.

Equation and parameters:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

a = 1/2 the longitude of the major axis
b = 1/2 the longitude of the minor axis
c = the distance from the centre to the focus
$a^2 + b^2 = c^2$

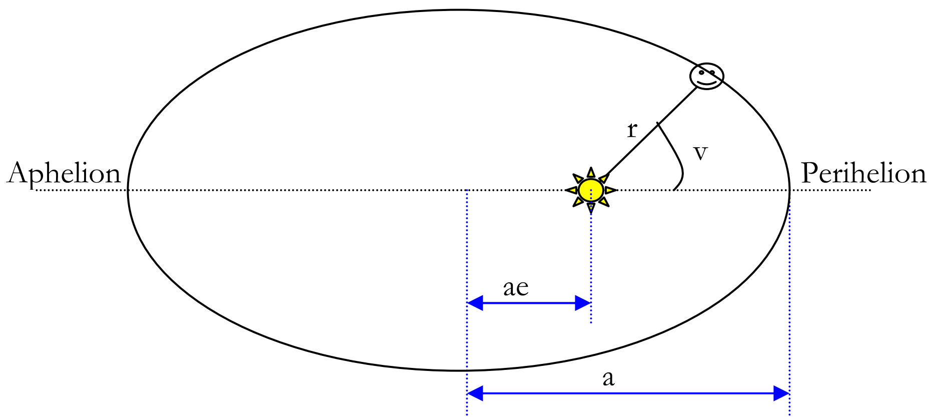
Eccentricity:: larger than 1

Chapter

2

Kinematics of planetary motion:

To determine the position of a planet in its orbit, it is necessary to know the orbit, that is, the major semi-axis of the orbit, (**a**) and the eccentricity (**e**). If we also wish to know where the planet is on a given date, **τ,** we will need an initial condition: normally we take the date on which it passed through the perihelion (or position closest to the Sun), as the date of reference $\boldsymbol{\tau_0}$ and the position of the perihelion as the origin of angles.



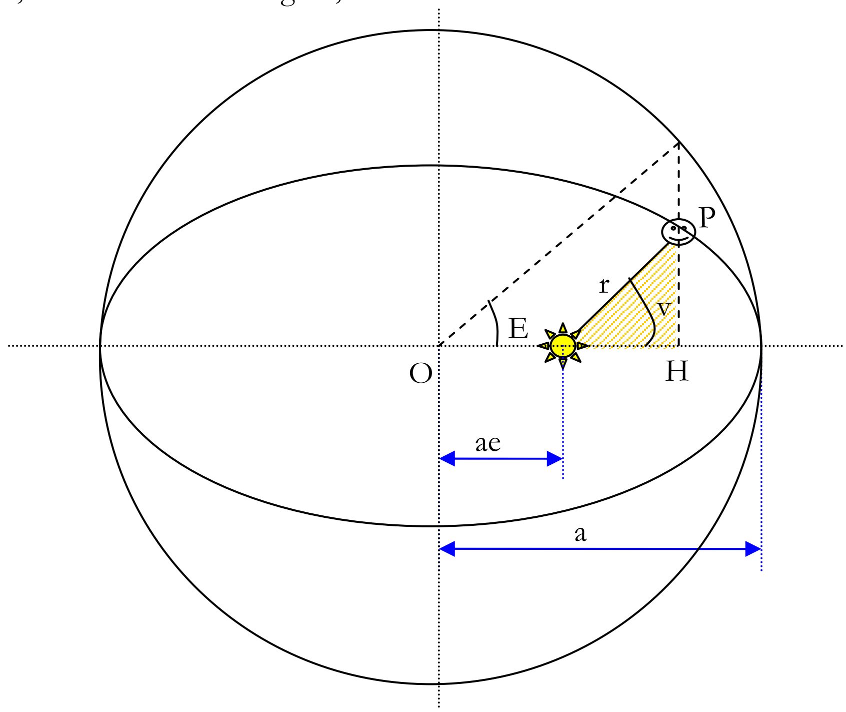
Once the geometry is defined, the kinematics of planetary movement are given in Kepler's Second Law: "the planets sweep out the same area in the same time no matter where in the orbit".

As a first approach, planets could be considered to move in circular orbits at constant velocity. The position of the planet is then defined by an angle, the *mean anomaly*, such that:

$$M(\tau) = \frac{2\pi}{T}(\tau - \tau_0)$$

where **T** represents the orbital period.

This "average" angle can be related to the *Eccentric Anomaly,* **E**, using Keppler's 2nd Law and some simple geometric relations between ellipse and circle, as indicated in the figure,



Kepler's Second Law states that:

$$\frac{\pi ab}{T} = \frac{SurfaceOHP}{\tau - \tau_0}$$

or,

$$M = \frac{2\pi}{T}(\tau - \tau_0) = \left(\frac{2}{ab}\right) SurfaceOHP$$

It can be shown from the figure that,

$$SurfaceOHP = \left(\frac{ab}{2}\right)(E - e\sin E)$$

so, *,* is obtained*:*

$$\boxed{M(\tau) = E(\tau) - esenE(\tau)}$$

This equation is known as ***Kepler's Equation*** and needs to be solved by numerical methods such as Newton's method (see Appendix).

Once **E** is obtained, the calculation of the *true anomaly,* **v**, and the distance Sun-Planet at this instant, **r**, is obtained in a direct manner, using the relationship between **E**, **r** and **v,** derived from the figure:

$$r(\tau) = a[1 - e \cdot \cos E(\tau)]$$

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

SUMMARY OF THE CHAPTER

1. Kepler's laws provide a good first order approximation to the kinematics of orbital motion.

2. Three fundamental angles (or "*anomalies*") are defined to describe the orbital motion.

   Mean Anomaly $M(\tau) = \frac{2\pi}{T}(\tau - \tau_0)$

   True Anomaly: V(τ) or angle between the planet and the perihelion.

   Eccentric Anomaly: E(τ)

3. These three angles are related by equations

$$M(\tau) = E(\tau) - esenE(\tau)$$

$$r(\tau) = a[1 - e \cdot \cos E(\tau)]$$

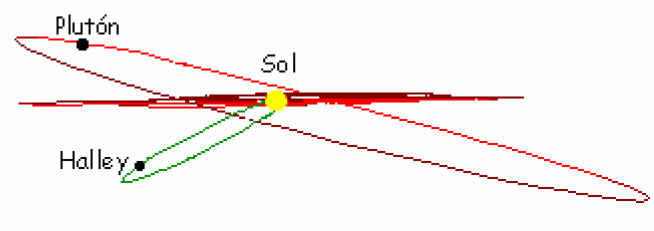
$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

Chapter

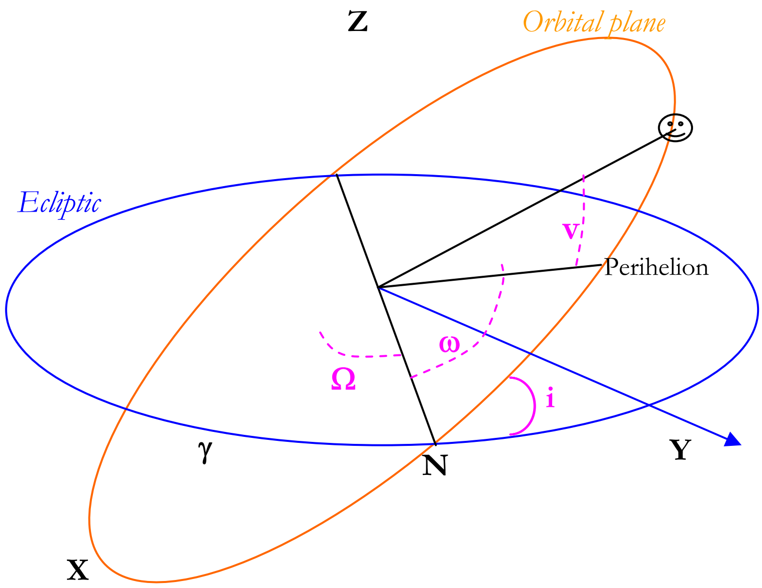
3

Orbital elements:

The orbits of the planets are not situated in the same plane, nor do they have the same orientation. It is necessary to introduce three geometric elements that define an orbital plane and the orientation of the orbit in the plane.



The orientation refers to a *Heliocentric Ecliptic System of Reference*, as seen in the figure:



and the new elements are:

- *inclination* of the orbital plane with respect to the ecliptic, **i**
- *ecliptic longitude of the ascending node*, **Ω**
- *argument of perihelion* or angle between the ascending node (N) and the direction of the perihelion, **ω**

Instead of the parameter **ω,** a new parameter is commonly used:

$$\tilde{\omega} = \omega + \Omega$$

called *longitude of perihelion.*

In summary, the parameters, **i,** ω **, Ω, a, e,** τ_0 define a unique orbit. These parameters are called *orbital elements.*

Orbital elements for all of the bodies in the Solar System (and also for satellites in orbit around the Earth) vary over time. The gravitational action of other bodies converts the two-body problem into a problem of N-bodies, which is not integrable, and needs to be solved numerically.

ORBITAL ELEMENTS OF THE PLANETS OF THE SOLAR SYSTEM						
Planet	**a** (**a.u.**)	**e**	**Ω** (**º**)	**ϖ** (**º**)	**i** (**º**)	$L(\tau) = M(\tau)+\varpi$ τ=4/June/2004 TU:00
Mercury	0.3871	0.2056	48.33	77.5	7.00	23.49
Venus	0.7233	0.0068	76.68	131.7	3.39	250.28
Earth	1.0000	0.0167	-	102.9	0.00	252.78
Mars	1.5237	0.0934	49.58	336.1	1.85	122.09
Jupiter	5.2026	0.0485	100.45	14.8	1.30	168.63
Saturn	9.5548	0.0555	113.66	94.3	2.49	104.32
Uranus	19.1817	0.0473	74.00	170.3	0.77	332.22
Neptune	30.0583	0.0086	131.78	67.7	1.77	314.50
Pluto	39.4817	0.2488	110.30	223.8	17.16	245.97

Chapter

4

Effect of Radiation Pressure

As shown in Manual 1 of this series, radiation pressure exerts a force that is inversely proportional to the square of distance, as gravity. This driving force is needed to sail the Solar System with solar sails, as those used in the application *"Sailing the Solar System"*. The two body problem is revisited in this Chapter but taking into account radiation pressure. As shown in Manual 1, solar radiation exerts a pressure given by[2]:

$$\boxed{\vec{F}_{P_\Theta} = \frac{1.12 \cdot 10^{17} \cdot S}{r^2}}$$

This thrust is braked by the gravitational force exerted by the Sun. As long as the sails maintain the same orientation with respect to the Sun, the problem is very easy to solve and can be integrated directly, as we shall see.

To begin with, notice that the dynamical equation [1] is modified to introduce the radiation force so,

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} - \vec{F}_{rad}$$

then, following the same procedure as in Chapter 2 (change to a mobile base and generation of two scalar equations from the vectorial equation) and substituting:

$$\vec{g} = -\frac{\mu}{r^2}\vec{u}_r$$

$$\vec{F}_{rad} = \frac{\kappa}{r^2} \cdot \zeta \cdot \vec{u}_r$$

[2] We consider the mass of the spacecraft as 100Kg (similar to that of the COSMOS-I mission) instead of 500Kg used in Manual I.

with, μ=G(M+m) and ζ=Scosφ, we obtain:

$$\ddot{r} - r\cdot\dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{\kappa}{r^2}\cdot\zeta = -\frac{(\mu-\kappa\cdot\zeta)}{r^2}$$

$$2\dot{r}\cdot\dot{\theta} + r\cdot\ddot{\theta} = 0 \Rightarrow r^2\cdot\dot{\theta} = h = cte$$

So, making the change of variable, $r = \frac{1}{u}$:

$$\ddot{r} - r\cdot\frac{h^2}{r^4} + \frac{(\mu-\kappa\cdot\zeta)}{r^2} = \ddot{r} - \frac{h^2}{r^3} + \frac{(\mu-\kappa\cdot\zeta)}{r^2} = 0$$

and,

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\cdot\dot{\theta}\cdot\frac{du}{d\theta} = -h\cdot\frac{du}{d\theta} \quad [3]$$

$$\frac{d^2r}{dt^2} = -h\cdot\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d^2\theta}\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\left(\frac{h}{r^2}\right) = -h^2u^2\frac{d^2u}{d\theta^2}$$

we obtain:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \left(\frac{\mu-\kappa\cdot\zeta}{h^2}\right)}$$

To bring in numbers, it is necessary to substitute the constants in the problem for realistic values. In the application *"Sailing the Solar System"*, we have considered that the spacecraft leaves a space station at L1- i.e. co-orbiting with the Earth around the Sun. Therefore, the angular momentum per unit of mass of the spacecraft will be very similar to that of the Earth:

$$h = 4.5\cdot 10^{19}\, cm^2 s^{-1}$$

If the mass of the spacecraft is similar to that of the prototype COSMOS-I (100 kg), then:

$$\kappa = 1.12\cdot 10^{17}\,\frac{dinas}{g}$$

and $\mu=1.33\cdot 10^{26}$ $g\cdot cm^3 s^{-2}$. So,

$$\boxed{\frac{d^2u}{d\theta^2} + u = 6.57\cdot 10^{-14} - 5.53\cdot 10^{-23}\zeta(cm^2) = \wp}$$

where $\wp$ is a constant for the fixed orientation of the Solar Sails.

Therefore, the radiation pressure will only have a significant effect on the trajectory if the surface of the ship is around 10^9cm^2 or 10^5m^2. The size proposed for the sail in the application (100,000m^2 is the surface of a circle of a radius of 178m), is derived from this estimate. The solution to the equation is:

$$\frac{1}{r} = A\cos(\theta - \theta_0) + \wp$$

with θ_0 and A constants of integration.

If we set the origin of the angles at the point of greatest proximity of the spacecraft to the Sun, then θ_0=0 and, in addition,

$$\frac{1}{r_{perihelio}} = A + \wp$$

Let us suppose that the distance from the spacecraft to the Sun in the perihelion is approximately equal to the average distance Sun-Earth($1.49\cdot10^{13}$cm). In this case, the value of the constant A will depend on the projected surface of the sails, ζ, as,

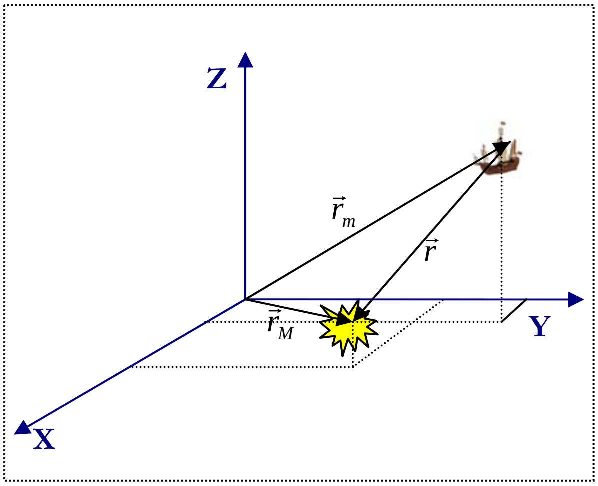
$$A = 8.33\cdot10^{-14} - 5.53\cdot10^{-19}\varsigma(m^2)$$

where A is given in cm^{-1}.

$$r = \frac{1/\wp}{1 + (A/\wp)\cdot\cos\theta}$$

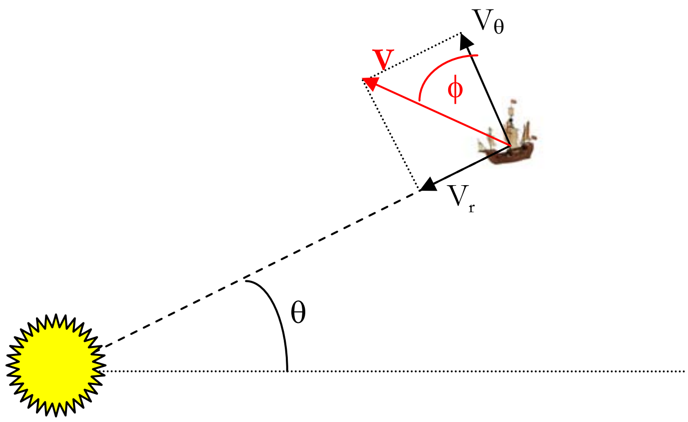
Thus orbit of a solar sail ship (that keeps the sails always oriented in the same direction in relation to the Sun) is an ellipse, although, for the same value of ε, the eccentricity of the orbit would be much larger than the purely gravitational orbital motion.

Finally, observe that vector $\vec{r}$ points to the Sun as, on defining our mobile base of vectors, we chose $\vec{r} = \vec{r}_M - \vec{r}_m$.



Components of the velocity vector:

At any moment, the components of velocity of the spacecraft (assuming ζ as a constant) will be:



***Azimutal Velocity*: V$_\theta$**

For a ship leaving the orbit of the Earth, V_θ is given in a direct way by the constant, h, the constant angular momentum

$$V_\theta = r\dot{\theta} = r\frac{h}{r^2} = \frac{h}{r} = \frac{4.5\cdot 10^{19}}{r}$$

if we wish to give r in astronomic units (see Manual I) and obtain V_θ in km/s,

$$V_\theta = \frac{30.20 km/s}{r(u.a.)}$$

Radial velocity: V_r

Radial velocity must be calculated from this equation [3],

$$V_r = \dot{r} = h \cdot A \cdot sen\theta \qquad [4]$$

substituting the constants h and A, we obtain:

$$V_r = \left[37.49km/s - 2.48 \cdot 10^{-4} km/s \cdot S(m^2) \cdot \cos\varphi\right] \cdot sen\theta$$

Notice that, depending on the effective surface of the sail, the radial velocity will fluctuate between very small or very large values. Another way of visualising this effect is by substituting sinθ for its value depending on r, in equation [4],

$$sen\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - \frac{1}{A^2}\left(\frac{1}{r} - \wp\right)^2} = \frac{1}{A}\sqrt{A^2 - \left(\frac{1}{r} - \wp\right)^2}$$

so that,

$$V_r = \frac{h \cdot A^2}{\sqrt{A^2 - \left(\frac{1}{r} - \wp\right)^2}}$$

General considerations on the trajectory and value of the energy constant:

The integration constant, A, is fixed by the energy per unit of mass in the orbit, as we have seen in Chapter 1. Therefore, it is usual to give V_r directly as a function of the energy per unit of mass, **ε**. This energy is the sum of the kinetic and potential energy:

$$\varepsilon = \varepsilon_c + \varepsilon_p = \frac{1}{2}(V_r^2 + V_\theta^2) + \left(-\frac{\mu}{r} + \frac{\kappa \cdot \varsigma}{r}\right) = \frac{1}{2}(V_r^2 + V_\theta^2) - \frac{\wp}{r} h^2$$

and,

$$V_r = \sqrt{2 \cdot \left[\varepsilon - \frac{V_\theta^2}{2} + \frac{\wp}{r} h^2\right]} = \sqrt{2\left[\varepsilon - \left(\frac{h^2}{2r^2} - \frac{\wp}{r} h^2\right)\right]}$$

Notice that V_r represents the modulus of the radial velocity, i.e., $V_r \geq 0$; this sets important constraints on the radical once ε is fixed. It is usual to define a function **Φ(r)** or ***Effective Potential***, such that,

$$\Phi(r) = h^2\left(\frac{1}{2r^2} - \frac{\wp}{r}\right)$$

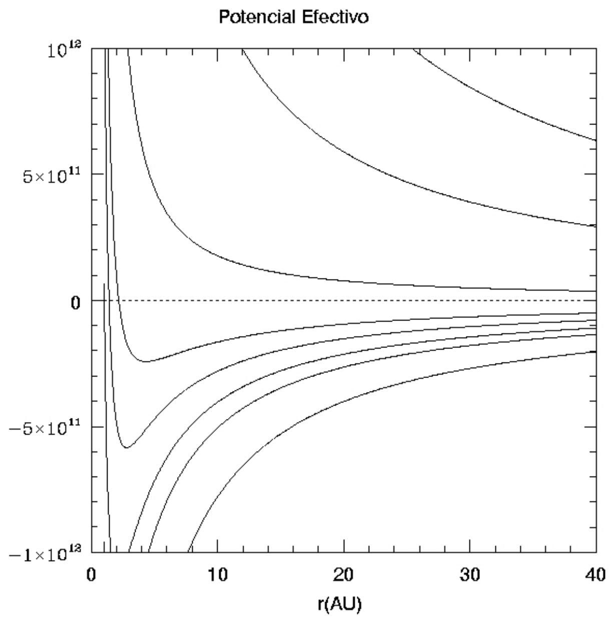
and substituting the constants,

$$\Phi(r) = \left(2.02 \cdot 10^{39}\, erg/g\right)\left(\frac{1}{2r^2} - \frac{\left(6.57 \cdot 10^{-14} - 5.02 \cdot 10^{-14}\xi\right)}{r}\right)$$

where, for convenience , we have introduced a new constant, ξ, to give us the fraction of the maximum possible surface total surface ξ_0=9.0792·$10^4$$m^2$ (equivalent to the surface of a circle with a radius of 170 m), covered by the sail,

$$\xi = \frac{S(m^2)\cos\varphi}{9.0792 \cdot 10^4}$$

The shape of the effective potential depends on the value of ξ , i.e., on the efficiency of the radiation pressure collector. If the radiation collector is very big, the spacecraft could escape the gravity of the Sun and leave the Solar System. To the contrary, if the collector is small, the spacecraft would be trapped in an orbit similar to that of the Space Port (and of the Earth). This can be seen in the figure:



The vertical axis of this figure indicates the value that should be adopted by the constant of energy **ε,** so that (ε-Φ) >0 and, therefore, we can keep the spacecraft in this position. The most negative curve corresponds to ξ=0.1 and the most positive to ξ=10. In the first case, the solar sail ship does not get enough thrust from solar radiation to escape the Space Port and will remain trapped in a nearby orbit. In contrast, if ξ=10, the spacecraft gets an enormous push from radiation and easily leaves the Port and the Solar System.

In practice, the value of ε is very small, $-4.48 \cdot 10^{12} \, erg/g$, at the Earth orbit and any reasonably sized sail (like the one proposed for the prototype Cosmos I) could not make it. In the application, we have used unrealistically high values for the surface to allow the students to play with the thrust of Solar Radiation. If they fill up a significant fraction of the total available surface (the circle at the begining of the application) they should be able to significantly modify V_r and thus the direction of the total velocity vector. This opens up the "launching windows" for solar sail ships. The objective of the application is to strengthen some basic concepts in Physics.

Notice that the "launching angle" or angle that the ship velocity makes with the vector Sun-Spacecraft is:

$$\tan \phi = \frac{V_r}{V_\theta}$$

Thus ϕ can be varied by modifying V_r that, in turn, can be modified by turning the sails (i.e. changing φ) since:

$$V_r(km/s) = 37.49 - 2.48 \cdot 10^{-4} S(m^2) \cos \varphi$$

The students are to go through this set of operations:

1. adjusting V_r with the graphic interface to launch the ship in the desired direction (get the resultant vector V aligned with the Space Port-Saturn direction)

2. the projected surface needed to get this resultant is calculated internally from:

$$S(m^2) \cos \varphi = \frac{37.49 - V_r}{2.48 \cdot 10^{-4}}$$

   and given to the student by the application.

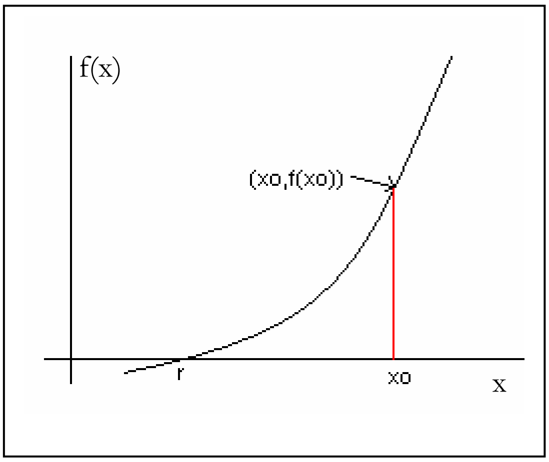
3. Later, students determine the projection angle φ using the surface of the sail as designed by them and the projected surface provided by the application.

Appendix

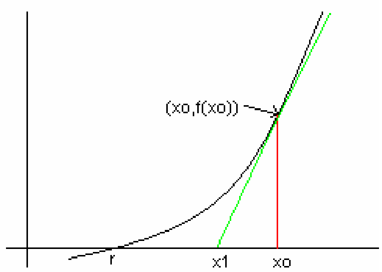
1

Newton's Method

Newton's Method is an iterative procedure to determine the **zero's** or roots, "**r**", of any equation $f(x) = 0$, providing that the function is "well-behaved". An initial guess value, "$\mathbf{x_0}$", is required and the first derivative of f(x), f'(x), should not change sign within the interval **($\mathbf{x_0}$, r)** to guarantee convergence.



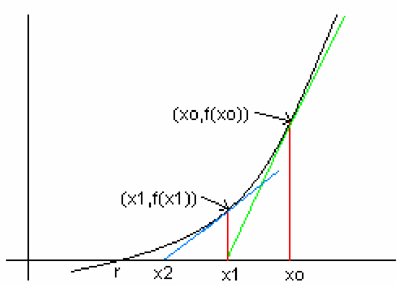
The straight line tangent to the curve at $(x_o, f(x_o))$ intersect the X-axis at "x_1", which is closer to the root "r" than "x_0". Thus,



$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

and, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

This procedure can be repeated as many times as needed so a new "x_2" may be defined, as in the figure, such that,



$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and, in general, the recurrence formula: $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}}$ (*)

Application to the resolution of Kepler's equation:

To find the solution to Kepler's equation is equivalent to find the zeros of the function f(E) such that:

$$f(E) = M - E + e \cdot senE$$

and making use of the Newton's method(*):

$$x_n = E_n$$
$$f(E_n) = M - E_n + e \cdot sen(E_n)$$
$$f'(E_n) = -1 + \cos(E_n)$$

then,

$$E_{n+1} = E_n - \frac{f(E_n)}{f^{'}(E_n)};$$
$$E_{n+1} = E_n - \frac{M - E_n + e \cdot senE_n}{-1 + \cos E_n}$$

and,

$$\boxed{E_{n+1} = E_n + \frac{M - E_n + e \cdot senE_n}{1 - \cos E_n}}$$

taking in the first iteration $E_0 = M$[3]

[3] Note: Angular values must be expressed in radians.